Functional matching of fshapes : a Γ-convergence result

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Résumé. Une forme fonctionnelle (fshape) est une surface sur laquelle est définie une fonction à valeurs réelles. Ce type de données, très courant en imagerie médicale, reste complexe à analyser d'un point de vue statistique.

Nous étudions ici le problème dit du "recallage fonctionel" (functional matching) entre deux fshapes. Il s'agit de trouver comment modifier le signal d'une des fshapes pour qu'elle ressemble le plus possible à l'autre fshape. C'est un problème variationnel qui consiste à minimiser une fonctionelle contenant un terme d'appariement entre les deux fshapes ainsi qu'une pénalité sur le signal (une norme L^2, H^1 ou BV). Nous présenterons des exemples numériques pour comparer ces différentes pénalités. Dans ces simulations, le problème continu est approché par un problème discret. Pour faire le lien entre ces deux problèmes, nous présenterons un résultat de Γ-convergence démontrant que les solutions aux problèmes de recallage fonctionel discret convergent bien vers la solution du problème continu (quand le nombre de points du problème discret augmente).

Mots-clés. Traitement d'images, Grande dimension

Abstract. A functional surface (fshape) is a surface on which a real function is defined. This type of data is very common in medical imaging but remain challenging to analyse on a statistical point of view.

We study here the functional matching problem between two fshapes. We want to find a signal on the source fshape making it as close as possible to the target fshape. This is a variational problem involving a functional composed of two terms. The first term is a distance between the fshapes. The second term is a penalty on the modified signal (L^2, H^1 or BV norm). We present some numerical simulations to compare the various penalty term. In the simulations, the continuous problem is approximated by a discrete problem. We prove a Γ-convergence result for the discrete matching energy towards the continuous-one. This results implies the convergence of the solution of the discrete problem toward the continuous one (as the number of point in the discrete problem increases).

Keywords. Image treatment, Large dimension

1 Introduction

This work informally presents the results of Charlier, Nardi & Trouvé (2016). No mathematical details are given here, the interested reader may find the rigorous framework in the paper. The aim of this short presentation is to give the statistical audience an idea of the tools developed to study the geometric and functional variability of textured surfaces.

New developments in non-invasive acquisition techniques such as Magnetic Resonance Imaging (MRI) or Optical Coherence Tomography (OCT) allow to get this kind of data usually after a segmentation step. A fshapes (X, f) is simply a smooth surface $X \subset R^3$ (possibly with boundary) on which a signal $f : X \to \mathbb{R}$ is defined. Typical example are given in Figure 1.

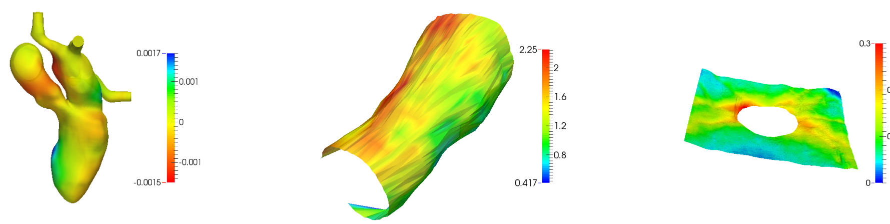

Figure 1: Three examples of fshapes. Left: blood pressure in a heart (courtesy C. Chnafa, S. Mendez et F. Nicoud, université de Montpellier). Center: thickness of a hippocampus (courtesy O. Colliot, INRIA). Right: thickness of retina (courtesy M. Suranic, S. Lee, F. Beg, Simon Fraser University)

In Charlier, Charon & Trouvé (2015), the authors introduce a new framework to study the matching problem for fshapes. Recall that a fshape (X, f) may be transformed by a metamorphosis (ϕ, ζ). A metamorphosis is a deformation with a geometric part given by a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ and a functional part $\zeta : X \to \mathbb{R}$. It acts on (X, f) in the following way :

$$(\phi, \zeta) \cdot (X, f) = (\phi(X), (f + \zeta) \circ \phi^{-1})$$

See Figure 2 for an illustration. Various theoretical results are given in Charlier, Charon & Trouvé (2015). A dicrete framework is also described and a software implementing this framework has been released (the fshapes toolkit that may be found at: `https://github.com/fshapes/fshapesTk`).

Figure 2: Fshape metamorphosis: geometry and signal change.

2 Functional matching problem

In this work we consider the "functional" matching problem between (X, f) and a target fshape (Y, g). We consider the following three energies:

$$E(f) = R(f, X) + V((X, f), (Y, g)))$$

where $R(f, X)$ can be defined as

$$\begin{aligned} R(f, X) &= \frac{1}{2}\|f\|^2_{L^2(X)} \quad (L^2\text{-model})\,, \\ R(f, X) &= \frac{1}{2}\|f\|^2_{H^1(X)} \quad (H^1\text{-model})\,, \\ R(f, X) &= \|f\|_{BV(X)} \quad (\text{BV-model})\,. \end{aligned}$$

and the attachment term V is defined by using the varifold theory. The L^2-model represents the model introduced in Charlier, Charon & Trouvé (2015).

We are interested here in the minimization problem with fixed geometry. This means that the optimization is made only with respect to the signal. In other words, the optimal fshape is supported on the initial surface X. We prove in particular an existence result for the optimal signal in the case of the BV and H^1-model. The existence result for the L^2-model is already proved in Charlier, Charon & Trouvé (2015) Theorem 6.

3 Discrete framework

We define a discrete version of the problem by approximating the surface X by a sequence of triangulations. The continuous problem can be then approximated by a sequence of discrete problems defined on some triangulations whose triangles's diameter goes to zero. Roughly speaking, smaller is the diameter of the triangles higher is the number of the vertices and, when the diameter goes to zero, the triangulation converges to the initial surface (with respect to the Hausdorff distance).

Concerning the definition of the discrete problems, the main issue is represented by the choice of the admissible triangulations. In fact, as X may have a boundary, the triangulation and the surface need not be one-to-one. In this paper, we decided to work with the class of triangulations that cover the surface and whose surpassing part has small area (Figure 3)

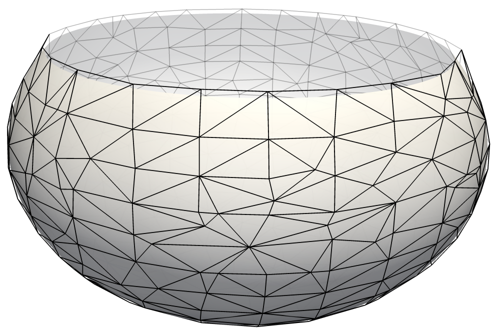

Figure 3: A surface X with boundary (solid grey) and an admissible triangulation (black lines).

Following Morvan & Thibert(2004), we point out the suitable properties a sequence of triangulations should verify to guarantee the convergence of the areas. To this end the angle between the normal to the triangulations and the respective (in the sense of the projection on the surface) normal to the surface has to go to zero.

4 Γ-convergence result

We must guarantee that the discrete solution is a "good" approximation of the continuous local minimum. This can be proved by the Γ-convergence theory, that is a natural notion of convergence of functionals allowing to justify the passage from discrete to continuous problems. In particular, in the case of minimization problems, the Γ-convergence guarantees also the convergence (in some sense) of the discrete minima towards the continuous-one. An introduction to the Γ convergence theory may be found in Jerrard & Sternberg(2009).

We prove a Γ-convergence result showing that the minimum of the discrete problem is close to the minimum of the continuous problem if the diameter of the triangles is small. The main issue to get such a result arises from the fact that the discrete problems and the continuous problem are not defined on the same geometric support. Now, in order to get the Γ-convergence result we need some hypothesis on the triangulations which guarantee the convergence of the areas. There is in fact a famous example called Schwartz's lantern proving that the area of the triangulations described above needs not to converge to the area of X.

For every penalty (L^2, H^1, BV), previous condition on the triangulations allows to prove the Γ-convergence result for the respective energy.

This kind of condition is involved in the numerical study of several problems defined on surfaces (e.g., Laplace-Beltrami operator). Then our result can be useful behind the matching model for fshapes.

5 Numerical experiments

Finally, we show some numerical examples to compare the different models. These examples actually point out that the BV-model strongly improves the matching result.

Bibliographie

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