ILL-DEFINED STATISTICAL TEST FOR FRACTIONAL INTEGRATION PARAMETER

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Abstract. In this paper we review recent developments in econometric from the point of view of statistical test theory. We show that some of the recent proposed statistical tests for the fractional integration parameter "d" in the context of $ARFIMA(p, d, q)$ models are fundamentally inappropriate.

Keywords. Fractional integration, Fractional unit root; LM test; Fractional Dickey-Fuller test; $ARFIMA$ models.

1 Introduction

As the most popular long memory model and a useful extension of the classical $ARIMA$ models, the fractionally integrated autoregressive moving average ($ARFIMA$) process has seen a considerable interest in the past three decades and has been widely applied in many fields like hydrology, economics and finance. The $ARFIMA$ process, introduced by Granger and Jojeux (1980) and Hosking (1981), generalizes the standard linear $ARIMA(p, d, q)$ model by permitting to the degree of integration d to be non-integer. Compared with the standard $ARMA$ and $ARIMA$ specifications, the $ARFIMA$ generalization provides a more flexible framework in modelling the long range dependence, where a special role is played by the fractional differencing parameter d whose precise determination is very important in applied work.

In recent years, an increasing effort has been made to establish reliable testing procedures to determine whether or not an observed time series is fractionally integrated. In particular, there has been a considerable interest in generalizing the familiar Dickey-Fuller test by taking into account the fractional integration order. It is well documented that the power of Dickey-Fuller [DF] type tests against alternatives of fractional integration is low (see Sowell (1990); Diebold and Rudebusch (1991); Hassler and Wolters (1994) ; Krämer (1998)). This motivated the development of powerful tests against fractional alternatives. Robinson (1991) pioneered an integration test constructed from the Lagrange Multiplier [LM] principle, which was proven by Robinson (1994) to be locally the most powerful under Gaussianity. The test has been further studied and modified by Agiakloglou and Newbold (1994), Tanaka (1999). Tanaka (1999) showed, through simulation experiments, that the LM tests have serious size distortion. Another serious criticism addressed to the

LM tests is that, by working under the null hypothesis, it does yield any direct information about the correct long-memory parameter d, when the null is rejected (Candelon, Gil Alana (2003)). Furthermore, the Lagrange multiplier (LM) test is based on the derivative of the log-likelihood. For the $ARFIMA(p,d,q)$ model

$$\Phi(B)(1-B)^d X_t = \Theta(B)\varepsilon_t, \tag{1.1}$$

where ε_t is white noise, the log-likelihood function can be written as

$$\log L = constant - \frac{T}{2}\log\sigma_\varepsilon^2 - \frac{1}{2\sigma_\varepsilon^2}\sum_{t=1}^{T}\left\{\frac{\Phi(B)}{\Theta(B)}(1-B)^d X_t\right\}^2. \tag{1.2}$$

Agiaklouglou and Newbold (1994) argues that, differentiating (1.2) with respect to d gives

$$\frac{\partial \log L}{\partial d} = -\frac{1}{\sigma_\varepsilon^2}\sum_{t=1}^{T}\left\{\frac{\Phi(B)}{\Theta(B)}(1-B)^d\log(1-B)X_t\right\}\varepsilon_t. \tag{1.3}$$

In fact, differenciating (1.2) with respect to d gives

$$\frac{\partial \log L}{\partial d} = -\frac{1}{\sigma_\varepsilon^2}\sum_{t=1}^{T}\left\{\frac{\Phi(B)}{\Theta(B)}\sum_{j=0}^{t-1}\left(\frac{\partial \mathbf{\Pi}_j(d)}{\partial d}\right)X_{t-j}\right\}\varepsilon_t, \tag{1.4}$$

where

$$\mathbf{\Pi}_j(d) = \frac{-d(-d+1)(-d+2)\cdots(-d+j-1)}{j!}$$

and

$$\frac{\partial \mathbf{\Pi}_j(d)}{\partial d} = \begin{cases} -1 & \text{if } j=1 \\ \frac{-\Pi_{i=1}^{j-1}(d+i)}{j!} + \sum_{i=1}^{j-1}\frac{-\Pi_{l=0}^{j-1}(d+l)}{(-d+i)j!} & \text{if } j \geq 2 \end{cases}.$$

All the ambiguities of the LM test, for fractional integration parameter come from the derivative (1.3).

More recently, Dolado et *al* (2002) introduced a fractional integration test (henceforth DGM test) based on an auxiliary regression for the null of unit root ($H_0 : d = 1$) against the alternative of fractional integration ($H_1 : d = d_1$, $d_1 < 1$). Their proposed test reduces to the standard Dickey-Fuller test when $d_1 = 0$ while under the null and when d_1 known, the statistic in the corresponding regression model depends on a fractional Brownian motion if $0 \leq d_1 < 0.5$. While the DGM test represents a useful generalization of the Dickey-Fuller test in the presence of a fractionally integrated alternative, it might give arbitrary conclusions when the null or the alternative hypotheses are misspecified, i.e. when the true d is not present neither in the null nor in the alternative. Indeed, through some simulation experiments we conduct, it may be seen (see Table 1 below) that the DGM test performs somewhat badly in the case where the parameter d is misspecified.

2 Fractional Dickey-Fuller testing: the DGM approach

2.1 Hypotheses and the auxiliary regression model

Dolado, Gonzalo, and Mayoral [DGM] (2002) introduced a test based on an auxiliary regression for the null of unit root against the alternative of fractional integration. The fractional Dickey-Fuller ($FD-F$) test considered by, in the basic framework, is described by the following. Let $\{y_t\}_{t=1}^{n}$ a series generated from the fractionally integrated process ($FI(d)$ in short) given by

$$(1-B)^d y_t = u_t,\ t \in \mathbb{Z}, \tag{2.1}$$

where $d \in \mathbb{R}$ is the true order of integration and, $\{u_t, t \in \mathbb{Z}\}$ is an iid innovation with mean zero and variance σ_u^2. For the data generating process (DGP) (2.1), DGM (2002) propose to test the following hypotheses,

$$H_0 : d = d_0 \text{ against } H_1 : d = d_1, \text{with } d_1 < d_0, \tag{2.2}$$

by means of the t statistic of the coefficient of $\Delta^{d_1} y_{t-1}$, where $\Delta = 1 - B$, in the ordinary least squares (OLS) regression

$$\Delta^{d_0} y_t = \rho \Delta^{d_1} y_{t-1} + \varepsilon_{t,}\ (t = 1, \cdots, n). \tag{2.3}$$

2.1.1 Unit root test against fractional alternatives and its asymptotic Properties

To study the performances of their procedure in terms of power and size, DGM (2002) consider only the particular case,

$$H_0 : d = 1 \text{ against } H_1 : d = d_1, \tag{2.7}$$

by means of the t-statistic of the coefficient $\Delta^{d_1} y_{t-1}$, in the ordinary least squares (OLS) regression

$$\Delta^1 y_t = \rho \Delta^{d_1} y_{t-1} + \varepsilon. \tag{2.8}$$

The t-ratio, $t_{\widehat{\rho}}(d_1)$, is given by

$$t_{\widehat{\rho}}(d_1) = \frac{\sqrt{n} \sum_{t=2}^{n} \Delta y_t \Delta^{d_1} y_{t-1}}{\sqrt{\sum_{t=2}^{n} \left(\Delta y_t - \widehat{\phi} \Delta^{d_1} y_{t-1} \right)^2 \sum_{t=2}^{n} \left(\Delta^{d_1} y_{t-1} \right)^2}}.$$

The implementation of DGM (2002) test would require tabulation of the percentiles of the functional of fractional Brownian motion, which imply that inference on the presence of unit root would be conditional on d_1. But given the well-known difficulties in estimating the order of fractional integration in finites samples, thus the test might suffer from misspecification. Under $H_0 : d = 1$, we have $Cov(\Delta^{d_0} y_t, \Delta^{d_1} y_{t-1}) = 0$ and under the alternative we have $Cov(\Delta^{d_0} y_t, \Delta^{d_1} y_{t-1}) = \sigma_\varepsilon^2(-1 + d_1) < 0$. Thus DGM build the decision rule as follows,

$$\begin{cases} H_0 : d = 1 \text{ is accepted} & \text{if } \rho = 0 \\ H_0 : d = d_1 \text{ is accepted} & \text{if } \rho < 0 \end{cases} \tag{2.9}$$

The hypotheses (2.7) based on the regression model (2.8) and the decision rule (2.9) is called by their authors "Fractional Dickey and Fuller Test".

2.1.2 Power and size of DGM's FDF test.

The test based on the hypotheses (2.7) and regression model (2.8) are useless in practice. The problem with the DGM type tests is that they are based on a choice of two possible orders of integration d_0 and d_1, of which the true order can be different either in the null or in the alternative. In fact, in the fractional integration case, there is a continuum of possible orders of integration. This would make the simple-versus-simple hypothesis invalid, particularly if the auxiliary regression model, used for the test, is based on the null and alternative. For instance, in the DGM test one of the following three cases holds:

- $d = d_0$,
- $d = d_1$,
- $d \neq d_0$ and $d \neq d_1$.

The third case causes serious troubles in practice, particularly, if the statistic of the test depends on null and alternative hypothesis. When $d_0 = 1$, in the first two cases, Dolado et al (2002) showed by means of a simulation study that their test procedure has a good performance in terms of power and level. For the third case, Dolado et al (2002) studied the effect of hypotheses misspecification by considering the deviations from the true value d_1 with size ± 0.1, ± 0.2 and ± 0.3. In the following; however, we replicate the simulation results of Dolado et al (2002) and present them more clearly by using a single table. We generate 1000 series from the data generating process (2.1) with sample size $n = 100$. The first column of Table 1 gives the true values of the parameter d while the second line shows the values of d_1 specified under the alternative. The first line gives the tabulated values by DGM (see Dolado et al [10], table X page 2003). The last line of Table 1 represents the performance of the DGM test in terms of level, i.e. the percentage of rejection of the null, when it is true (α), while the main diagonal represents the performance of the DGM test in terms of power i.e. the percentage of acceptance of the alternative hypothesis when it is true, $(1 - \beta)$. α and β are respectively the type I and the type II errors, defined by

$$\alpha = P(\text{reject } H_0 | d = 1) \quad \text{and} \quad \beta = P(\text{reject } H_1 | d = d_1).$$

The other values in the table are the percentage of acceptance of the alternative hypothesis when both the null and alternative are false i.e. when the value of d is wrongly specified. In fact, these values represent another type of errors, namely

$$P_{d \neq d_1} \left(\text{Accept } H_1 | d \neq 1 \text{ and } d \neq d_1 \right).$$

When performing a test one may arrive at the correct decision, or one may commit one of two errors: rejecting the null hypothesis when it is true (type I error, or error of the first kind) or accepting it when it is false (type II error or error of the second kind). In statistical testing theory, there is no place for type III error (or error of the third kind). This anomaly is the consequence of the choice of inappropriate auxiliary regression model, which depends on the null and alternative. From Table 1, it may be easily observed that when the true d is well specified, the DGM test has a good performance in terms of power and level. However, in the case where the true value of $d \in [0,1] - \{1, d_1\}$, the conclusions of the test are somewhat arbitrary. For example, when $d = 0.3$, the percentage of acceptance of the alternative is equal 100% regardless of the alternative hypothesis. In other word, if the process y_t, is fractionally integrated of order $d = 0.3$ (i.e. stationary stationary process), the table 1, show that for $H_0 : d = 1$ against $H_1 : d = 0.7$, we have

$$P_{d=0.3} \left(\text{Accept } H_1 : d = 0.7 | d \neq 1 \text{ and } d \neq d_1 \right) = 1.$$

This example shows clearly that the risk to specify the stationary process as a nonstationary process is high.

Table 1: Performance of the DGM test in terms of size (i.e. when $d = 1$), power (i.e. when $d = d_1$) and when the value of d is wrongly specified (i.e. $d \neq 1$ and $d \neq d_1$)

Tabulated values $\alpha = 5\%$	−1.95	−1.87	−1.84	−1.82	−1.81	−1.75	−1.65	−1.65	−1.65	−1.65
$H_1 : d = d_1$ / True value d	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	100	100	100	100	100	100	100	100	100	100
0.1	100	100	100	100	100	100	100	100	100	100
0.2	100	100	100	100	100	100	100	100	100	100
0.3	100	100	100	100	100	100	100	100	100	99.9
0.4	100	100	100	100	100	100	100	100	99.9	99.6
0.5	100	100	100	100	100	100	100	100	100	100
0.6	89.5	99.5	100	100	100	100	100	100	100	100
0.7	65.2	83.1	90.6	95.8	97.1	96.7	95.7	92.9	88.8	82.8
0.8	33.8	47.2	56.9	65.1	71.2	72.7	74.3	69.9	65.2	58.3
0.9	14.7	18.4	21.2	23	24.1	26.8	29.3	29	27	25.2
$H_0 : d = 1$	4	4.5	4.8	5.4	4.8	4.9	5.2	5.4	5.3	5.5

The DGM (2002) test presents an analogy with the original Dickey-Fuller test, but cannot be considered as a generalization of the familiar Dickey-Fuller test in the sense that the conventional $I(1)$ vs $I(0)$ framework is recovered (for the DGM test the conventional framework is recovered only if $d_0 = 1$ and $d_1 = 0$).

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