Estimation in functional convolution model

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Résumé. L'objectif de cet article est de proposer un estimateur de la fonction inconnue dans le Modèle Fonctionnel de Convolution (FCVM), qui étudie la relation entre une covariable fonctionnelle $X(t)$ et une sortie fonctionnelle $Y(t)$ au travers de l'équation suivante $Y(t) = \int_0^t \theta(s)X(t-s)ds + \epsilon(t)$, où θ est la fonction à estimer et ϵ est un bruit fonctionnel additif. Il permet d'étudier l'influence de l'historique de X sur $Y(t)$.

Nous utilisons la Transformée de Fourier Continue (CFT) pour définir un estimateur de θ. La transformation du modèle de convolution donne le Modèle Fonctionnel de Concurrence (FCCM), dans le domaine des fréquences, à savoir $\mathcal{Y}(\xi) = \beta(\xi)\mathcal{X}(\xi) + \varepsilon(\xi)$. Afin d'estimer la fonction inconnue β, nous avons étendu la méthode ridge de régression classique au cadre de données fonctionnelles. Nous établissons des propriétés de consistence des estimateurs proposés et illustrons nos résultats avec des simulations.

Mots-clés. Données Fonctionnelles, Modèle de Convolution; Modèle de Concurrence; Transformée de Fourier.

Abstract. The aim of the paper is to propose an estimator of the unknown function in the Functional Convolution Model (FCVM), which studies the relationship between a functional covariate $X(t)$ and a functional output $Y(t)$ through the following equation $Y(t) = \int_0^t \theta(s)X(t-s)ds + \epsilon(t)$, where θ is the function to be estimated and ϵ is an additional functional noise. It allows to study the influence of the history of X on $Y(t)$.

We use the Continuous Fourier Transform (CFT) to define an estimator of θ. The transformation of the convolution model results in the Functional Concurrent Model (FCCM) associated, in the frequency domain, namely $\mathcal{Y}(\xi) = \beta(\xi)\mathcal{X}(\xi) + \varepsilon(\xi)$. In order to estimate the unknown function β, we extended the classical ridge regression method to the functional data framework. We establish consistency properties of the proposed estimators and illustrate our results with some simulations.

Keywords. Functional data; Convolution model; Concurrent model; Fourier transform.

1 Introduction

Functional Data Analysis (FDA) proposes very good tools to handle data that are functions of some covariate (e.g. time, when dealing with longitudinal data). These tools may allow a better modelling of complex relationships than classical multivariate data analysis would do, as noticed by Ramsay and Silverman [2005, Ch. 1], Yao et al. [2005], among others.

There are several models in FDA to study the relationship between two variables. In particular in this paper we are interested in the Functional Convolution Model (FCVM)

$$Y(t) = \int_0^t \theta(s)\, X(t-s)ds + \epsilon(t), \tag{1}$$

where $t \geq 0$, θ is the function to be estimated, X, Y are random functions and ϵ is a noise random function. This model allows to study the influence of the history of X on $Y(t)$. It was derived from Malfait and Ramsay [2003] by requiring that the function θ remains the same for each time t (i.e it only depends on s).

In this paper we propose an estimator of θ for a given i.i.d. sample $(X_i, Y_i)_{i\in\{1,\cdots,n\}}$ of the random functions X and Y. This question has been poorly addressed in the literature. Asencio et al. [2014] studies a similar problem. They proposed to estimate θ by approximating its projection into finite-dimensional Spline subspaces. Another related problem is the multichannel deconvolution problem (see e.g. Pensky and Sapatinas [2010]), where the periodic case is studied.

The main idea of our approach is to use the Continuous Fourier Transform (CFT) to transform the problem into its equivalent in the frequency domain. It yields the following Functional Concurrent Model (FCCM):

$$\mathcal{Y}(\xi) = \beta(\xi)\, \mathcal{X}(\xi) + \varepsilon(\xi), \tag{2}$$

where $\xi \in \mathbb{R}$, β is an unknown function to be estimated, $\mathcal{X} := \mathcal{F}(X)$, $\mathcal{Y} := \mathcal{F}(Y)$ are Fourier transforms of X and Y, and $\varepsilon := \mathcal{F}(\epsilon)$ is an additive functional noise. Once there, we use an estimator of β for the FCCM, and then we come back to the time domain through the Inverse Continuous Fourier Transform (ICFT).

Some related models to the FCCM have already been discussed by several authors. For instance Hastie and Tibshirani [1993] has proposed a generalization of the FCCM called 'varying coefficient model'. Until now many methods have been developed to estimate the unknown smooth regression function β, for instance by local maximum likelihood estimation (see e.g. Dreesman and Tutz [2001]), or by local polynomial smoothing (see e.g. Fan et al. [2003]). These methods use techniques from multivariate data analysis, as noticed by Ramsay and Silverman [2005, p. 259]. We propose an approach that avoids projecting into a finite dimensional sub-space and allows to work with the entire Fourier transforms in the Frequency domain. It is an extension of the Ridge Regression method to the functional case.

We prove the consistency of our estimators and show some simulation results.

2 Model and Estimator

2.1 Notations

For $r = 1, 2$, $L^r := L^r(\mathbb{R}, \mathbb{C})$, is the space of complex valued functions defined on $\mathbb{R}$, with the L^r-norm, $\|f\|_{L^r} := \left[\int_{\mathbb{R}} |f(x)|^r dx\right]^{1/r}$. $C_0 := C_0(\mathbb{R}, \mathbb{C})$ is the space of complex valued continuous functions which vanish at infinity, with the supremum norm $\|f\|_{C_0} := \sup_{x\in\mathbb{R}} |f(x)|$. Finally the support of the continuous function $f : \mathbb{R} \to \mathbb{C}$ is the set $supp(f) := \{t \in \mathbb{R} : |f(t)| \neq 0\}$. Besides we define the boundary of a set S, as $\partial(S) := \overline{S} \setminus int(S)$, where $\overline{S}$ is the closure of S and $int(S)$ is its interior.

2.2 General Hypotheses of the FCVM

We consider general hypotheses that will be used along the paper.

$(HA1_{FCVM})$ X, ϵ are independent $L^1 \cap L^2$ valued random functions such that $\mathbb{E}(\epsilon) = \mathbb{E}(X) = 0$ and for every $t < 0$, $\epsilon(t) = X(t) \equiv 0$.

$(HA2_{FCVM})$ $\theta \in L^2$ and for every $t < 0$, $\theta(t) \equiv 0$.

$(HA3_{FCVM})$ $\mathbb{E}(\|\epsilon\|^2_{L_1}), \mathbb{E}(\|X\|^2_{L_1}) < +\infty$, $\mathbb{E}(\|\epsilon\|^2_{L^2}), \mathbb{E}(\|X\|^2_{L^2}) < +\infty$.

The Laplace convolution integrating between 0 and t is equivalent to the Fourier one because for every $t < 0$, $\theta(t) = X(t) \equiv 0$, $(HA1_{FCVM})$ and $(HA2_{FCVM})$.

2.3 Functional Fourier Deconvolution Estimator (FFDE)

Before defining the estimator of θ, we transform the problem to the frequency domain (see (2)) and then we estimate β with the **Functional Ridge Regression Estimator (FRRE)** which is an extension of the Ridge Regularization method (Hoerl [1962]) that deals with ill-posed problems in the classical linear regression.

Let $(X_i, Y_i)_{i=1,\cdots,n}$ be an i.i.d sample of FCVM (1) and a regularization parameter $\lambda_n > 0$. We define the estimator of β in the FCCM (2) as follows

$$\hat{\beta}_n := \frac{\frac{1}{n}\sum_{i=1}^n \mathcal{Y}_i\, \mathcal{X}_i^*}{\frac{1}{n}\sum_{i=1}^n |\mathcal{X}_i|^2 \;+\; \frac{\lambda_n}{n}}, \tag{3}$$

where $*$ is the complex conjugate. Then the estimator of θ in (1) is defined by

$$\hat{\theta}_n := \mathcal{F}^{-1}(\hat{\beta}_n). \tag{4}$$

3 Consistency of the FRRE and the FFDE

We prove the consistency of the estimators in the following theorem:

Theorem 1 *Let $(X_i, Y_i)_{i\geq 1}$ be i.i.d. realizations of FCVM which satisfies the general hypotheses $(HA1_{FCVM})$-$(HA3_{FCVM})$ and*

(A1) $\overline{supp(|\beta|)} \subseteq \overline{supp(\mathbb{E}[|\mathcal{X}|])}$,

(A2) $(\lambda_n)_{n\geq 1} \subset \mathbb{R}^+$ *is such that* $\frac{\lambda_n}{n} \to 0$ *and* $\frac{\sqrt{n}}{\lambda_n} \to 0$ *as* $n \to +\infty$.

Then

$$\lim_{n\to+\infty} \|\hat{\theta}_n - \theta\|_{L^2} = \lim_{n\to+\infty} \|\hat{\beta}_n - \beta\|_{L^2} = 0 \quad \textit{in probability.} \tag{5}$$

Hypothesis *(A1)* is about the intervals where we can estimate β. If the curve $\mathcal{X}$ is zero in some interval then we cannot estimate β in that interval because $\mathcal{Y}$ is equal to the noise and does not contain any information about β. **Hypothesis** *(A2)* is classic in the context of the Ridge regression.

Some convergence rates of both estimators have also been obtained under additional assumptions. We now illustrate these results with simulations.

4 Simulations

4.1 Description of the Simulation

We consider samples of $n = 150$ and $n = 500$ realizations. The curves were simulated over the interval $[0, 1]$. We have $p = 200$ equispaced observation times, that is for $j \in \{0, \cdots, 199\}$, $t_j := j * 0.005$.

The output curves were generated from $Y_i(t) = \theta * X_i(t) + \epsilon_i(t)$, where $\theta(t) = 1.63\sin((4 - 1/2)\pi t)$ when $t \in [0, 0.5]$, and zero otherwise. X_i and ϵ_i are independent Brownian Bridges over $[0, 0.5]$ and zero function over $[0.5, 1]$. The noise ϵ_i is multiplied by a factor in such a way that the signal-to-noise ratio (SNR) is equal to 4.

4.2 Graphical results

Figure 1 an example of simulated curves X_i and Y_i. Figure 2 shows the estimation of θ in the FCVM for $n = 150$ and $n = 500$. We use the Predictive Cross Validation (PCV) to choose the regularization parameter λ_n. The computing times of the estimators are respectively $0.677sec$ and $2.410sec$ for $n = 150$ and $n = 500$. This computation time includes the optimization of the PCV criteria to choose λ_n.

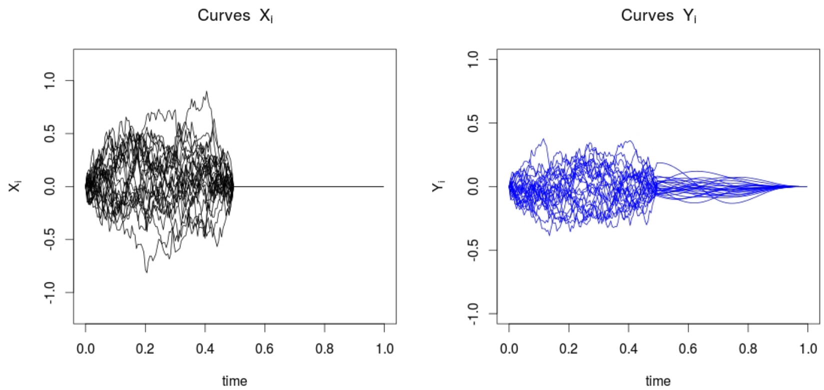

Figure 1: Simulated curves X_i and Y_i.

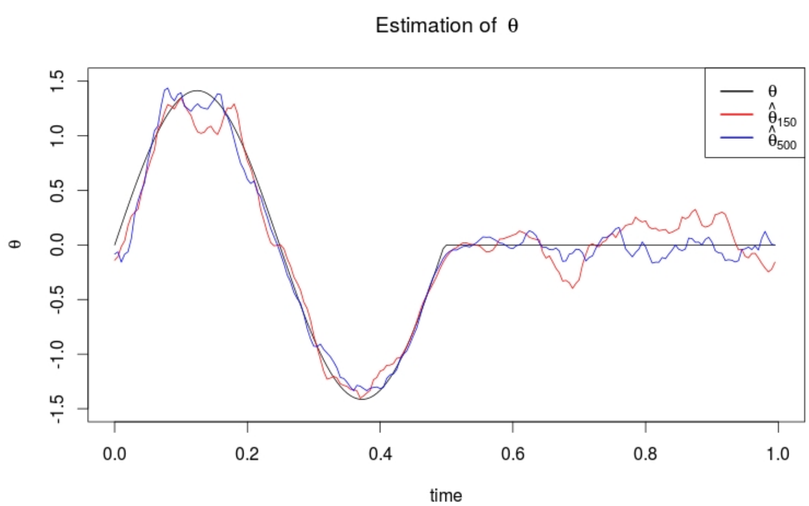

Figure 2: Estimation of θ with $n = 150$ and $n = 500$ observations. The PCV optimal parameters are $\lambda_{150} = 757.25$ and $\lambda_{500} = 902.33$ respectively.

5 Conclusions

In this paper we have shown the probability convergence of the FRRE and FFDE estimators for the FCCM and FCVM. In this approach we do not need to use finite dimensional subspaces to define the estimators, contrary to many other approaches. In applications the Fast Fourier Transform allows us to compute the estimator quickly, faster than other estimation methods.

This work is a first step to handle high-throughput plant phenotyping data. Model FCVM (1) may be used to finely model the influence of environmental variables (temperature, light, etc.) on leaf elongation rate.

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